CHAPTER 7

PROPERTIES OF PLANE AREAS

7.1 Introduction

The carrying capacity of a structural member depends on three main factors; the strength of the material used, the type of loading and the properties of the member's cross-section. The first two factors will be considered in later chapters dealing with stresses, while in the present chapter the third factor is dealt with.

The different properties of sections which are likely to be determined in the problems of structural strength are :

(1) Cross-sectional area.

(2) Position of centre of area or centroid.

(3) Moment of inertia or second moment of area.

(4) Polar moment of inertia.

(5) Radius of gyration.

(6) Product of inertia.

(7) Principal axes of inertia.

It should be remembered that for a particular problem it may not be necessary to calculate but a few of these properties.

The cross-sectional area, A, needs no description and may be easily calculated for most structural shapes.

7.2 Centre of area or centroid

This is defined as the point 0 in the plane at which the area may be assumed to be concentrated to cause the same moment about an axis in the plane as the distributed area. Thus, referring to Fig. 7.1, the co-ordinates of the centroid $(\bar{x}, \bar{y})$ with respect to the rectangular axes y and x are given by :

$$\bar{x} = \frac{\int x dA}{A} \qquad \text{... 7.1 a}$$

$$\bar{y} = \frac{\int y dA}{A} \qquad \text{... 7.1 b}$$

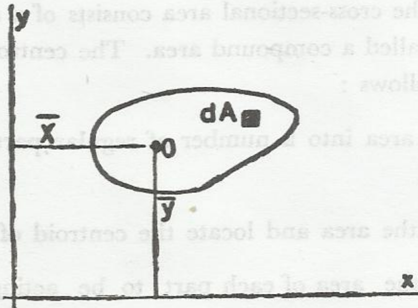

Fig. 7.1

where A is the total area, and $\int ydA$ and $\int xdA$ are the first moments of area or the statical moments about the x and y axes respectively. These are commonly denoted by S_y and S_x. Thus,

$$S_x = \int ydA \qquad \text{... 7.2 a}$$

$$S_y = \int xdA \qquad \text{... 7.2 b}$$

It follows from the definition that if an area has an axis of symmetry, the centroid lies on it and if, further, it has two or more axes of symmetry, it is at their point of intersection. This result helps in detecting the centroids of many areas as may be seen from Fig. 7.2.

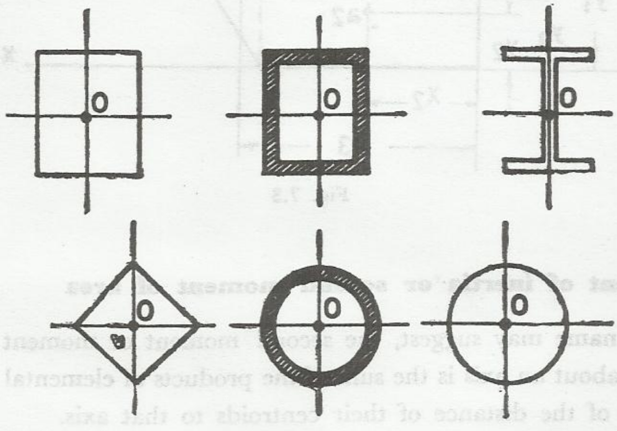

Fig. 7.2

The centroids of regular areas such as the triangle, the semi-circle, the parabola, etc. are best found by calculus and the student is advised to memorize them. These are given in Appendix 1.

In many cases, the cross-sectional area consists of a number of regular parts and is hence called a compound area. The centroid of a compound area is found as follows :

(a) Divide the area into a number of regular parts whose centroids are known.

(b) Determine the area and locate the centroid of each part.

(c) Assuming the area of each part to be acting at its centroid, take moments about two convenient rectangular axes. The centroid of the compound area shown in Fig. 7.3 has thus the co-ordinates $\bar{x},\bar{y}$ where $\bar{x}$ and $\bar{y}$ are given by :

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{A}$$

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{A}$$

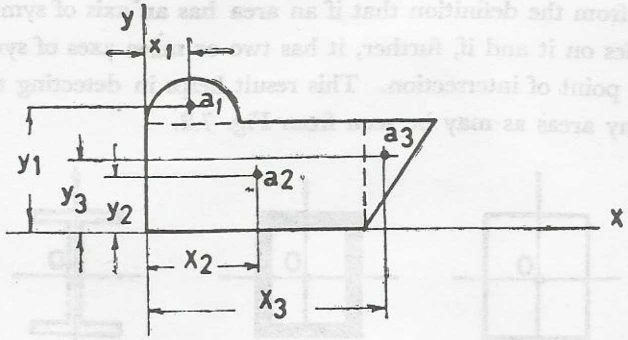

Fig. 7.3

7·3 Moment of inertia or second moment of area

As the name may suggest, the second moment or moment of inertia of an area about an axis is the sum of the products of elemental areas and the square of the distance of their centroids to that axis.

Thus, referring to Fig. 7.4, the moment of inertia which is usually denoted by I may be expressed mathematically as :

$$I_x = \int y^2 dA \qquad \dots 7.3$$

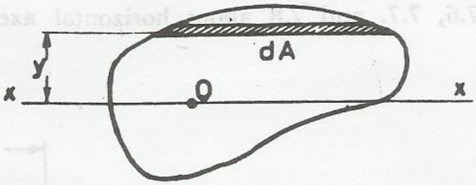

Fig. 7.4

It follows from the definition that the moment of inertia is always positive and its unit is length to the fourth power; cm^4 or $m.^4$. The moments of inertia of the common shapes given in Appendix 1 should be memorized as they are frequently used.

7.4 Theorem of parallel axes

The moment of inertia about any axis is equal to the moment of inertia about a parallel axis through the centroid plus the product of the total area and the square of the distance between the two axes. Referring to Fig. 7.5, this could be expressed mathematically as :

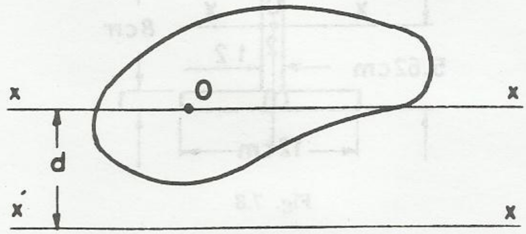

Fig. 7.5

$$I'_x = I_x + Ad^2 \qquad \dots 7.4$$

In transferring a moment of inertia between two axes, neither of which is through the centroid, it is necessary first to find the moment of inertia about the centroidal axis then transfer it to the required axis by using equation 7.4 twice. It is seen that I about a centroidal axis is smaller than that about any other parallel axis.

Examples 7.1-7.3 Determine the moments of inertia of the areas shown in Figs. 7.6, 7.7. and 7.8 about horizontal axes through their centroids.

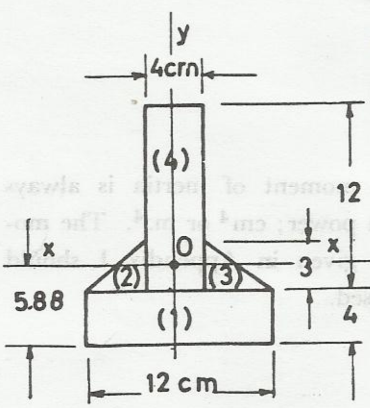

Fig. 7.6

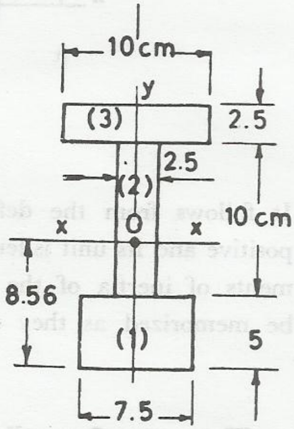

Fig. 7.7

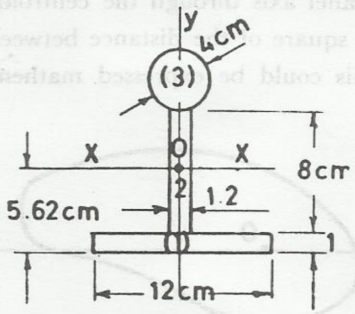

Fig. 7.8

Solution to example 7.1 :

Referring to Fig. 7.6, since the area is symmetrical about the y-axis, the centroid must lie on it, and it remains to calculate its distance from a convenient axis such as that at the lower edge of the section to completely locate the centroid. In order to do so, the area is divided into triangles and rectangles. For convenience, the computations are given in a tabulated form as shown in Table 7.1. The areas of individual parts are tabulated in column (2), the co-ordinates of the centroids of the elements are tabulated in column (3), and the first moments of area, Ay, are tabulated in column (4). The centroid of the total area is obtained by dividing the sum of the terms in column (4) by the sum of the terms in column (2). Thus,

$$\bar{y} = \frac{636}{108} = 5.88 \text{ cm.}$$

The moment of inertia of the total area about the x-axis will be obtained as the sum of the moments of inertia of the various elements about this axis; the moment of inertia of each element being given by equation 7.4. The terms I_x and Ad^2 for all the elements are given in columns (5) and (6) respectively. The moment of inertia of the entire area about the x-axis is equal to the sum of all the terms in columns (5) and (6). Thus,

$$I_x = 646 + 1553 = 2199 \text{ cm}^4$$

Element	A cm².	y cm.	Ay cm.³	I cm⁴.	Ad² cm.⁴
1	48	2	96	64	728
2	6	5	30	3	5
3	6	5	30	3	5
4	48	10	480	576	815
Total	108		636	646	1553

Table 7.1

Solution to Examples 7.2 & 7.3 :

The calculations for the other two areas in Figs. 7.7. and 7.8 are summerized in Tables 7.2 and 7.3 respectively.

Element	A cm.²	y cm.	Ay cm.³	I cm.⁴	Ad² cm.⁴
1	37.5	2.5	94	78	1380
2	25	10	250	208	52
3	25	16.25	406	13	1482
Total	87.5		750	299	2914

Table 7.2

$$\bar{y} = \frac{750}{87.5} = 8.56 \text{ cm.}$$

$$I_x = 299 + 2914 = 3213 \text{ cm.}^4$$

Element	A cm.^{2}	y cm.	Ay cm.^{3}	I cm.^{4}	Ad^2 cm.^{4}
1	12	0.5	6	1	315
2	9.6	5	48	51.2	3.7
3	12.6	11	138.6	12.6	364
Total	34.2		192.6	64.8	682.7

Table 7.3

$$\bar{y} = \frac{192.6}{34.2} = 5.62 \text{ cm.}$$

$$I_x = 64.8 + 682.7 = 7.47.5 \text{ cm.}^4$$

It should be noticed that if the area has more than one axis of symmetry, the calculations are simpler as the position of the centroid is known to be at the point of intersection of these axes. Also, if the moment of inertia of one of the compound areas about a vertical centroidal axis is required, calculations may still be conducted in the manner outlined above.

The following example will illustrate these points.

Example 7.4 Find the moments of inertia for the section shown in Fig. 7.9 about the horizontal and vertical axes through the centroid.

Solution : The centroid is known to be at the point of intersection of the two axes of symmetry. The moment of inertia of the whole area about the x-axis will be obtained as the sum of the moments of inertia of a rectangle about its centroidal axis and the moments of inertia of four triangles about an axis through their base. Thus,

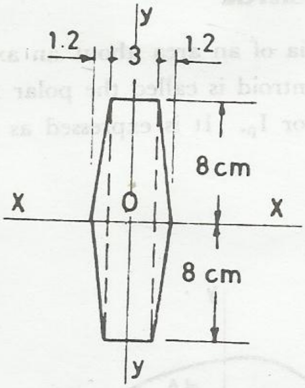

Fig. 7 9

$$I_x = \frac{3 \times 16^3}{12} + 4\left(\frac{1.2 \times 8^3}{12}\right) = 1228.8 \text{ cm.}^4$$

The moment of inertia of the whole area about the y-axis will be obtained as the sum of the moment of inertia of a rectangle about its centroidal axis and the moments of inertia of two triangles about an axis parallel to their centroidal axes and at a distance d,

$$d = 1.5 + 1.2/3 = 1.9 \text{ cm.}$$

Thus, $I_y = \frac{16 \times 3^3}{12} + 2\left(\frac{16 \times 1.2^3}{36} + \frac{16 \times 1.2 \times 1.9^2}{2}\right)$

$$= 106.4 \text{ cm.}^4$$

It should be noticed that a beam with a cross-section such as that shown in Fig. 7.9 may be used with the longer dimension either vertical or horizontal, and it will offer more resistance to bending when placed in the former position. This is because the moment of inertia about the x-axis is larger than that about the y-axis as indicated by the above calculations.

The student is advised to solve the previous examples independently and check his results against those given. Further, he can attempt to calculate the moments of inertia about the y-axis.

7.5 Polar moment of inertia

The moment of inertia of an area about an axis perpendicular to its plane and through its centroid is called the polar moment of inertia and is usually denoted by J or I_p. It is expressed as :

$$J = \int r^2 dA \qquad \ldots 7.5$$

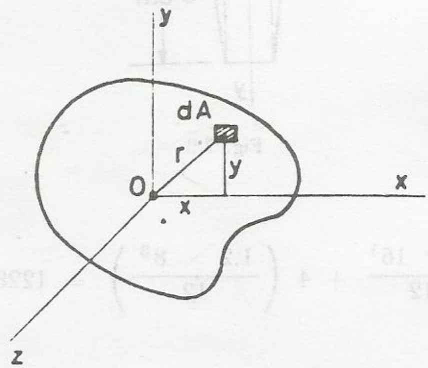

Fig. 7.10

Referring to Fig. 7.10, $r^2 = x^2 + y^2$ and from equation 7.5.

$$J = \int (x^2 + y^2)\, dA = I_x + I_y \qquad \ldots 7.6$$

Equation 7.6 shows that the sum of the moments of inertia about any two rectangular centroidal axes is constant. It follows that if the moment of inertia about one of these axes is a maximum, the moment about the other must be a minimum. The polar moment of inertia of a circular section is frequently used in problems dealing with torsion of shafts with circular cross-section (chapter 9). For this section, it is known from symmetry that $I_x = I_y$. Hence from equation 7.5.

$$J = \frac{\pi r^4}{4} + \frac{\pi r^4}{4} = \frac{\pi r^4}{2}$$

7.6 Radius of gyration

The radius of gyration of a section is the distance from the inertia axis that the entire area may be assumed to be concentrated in order to give the same moments of inertia. Thus by definition, the radius of gyration which is usually denoted by i may be expressed as :

$$I = Ai^2 \qquad \text{... 7.7}$$

$$\text{or} \quad i = \sqrt{\frac{I}{A}} \qquad \text{... 7.8}$$

It is seen that the point where the area is assumed to be concentrated is not the same as the centroid. It is also different for each inertia axis chosen.

$$i_x = \sqrt{\frac{I_x}{A}} \qquad \text{... 7.9 a}$$

$$i_y = \sqrt{\frac{I_y}{A}} \qquad \text{... 7.9 b}$$

This area property is of particular importance in regard to problems dealing with buckling of columns.

7.7 Product of inertia

The product of inertia about two rectangular axes x and y is defined as the sum of the products of elemental areas and the co-ordinates of their centroids to the reference axes. Thus, referring to Fig. 7.11, the product of inertia which is usually denoted by I_{xy} may be expressed mathematically as :

$$I_{xy} = \int xydA \qquad \text{... 7.10}$$

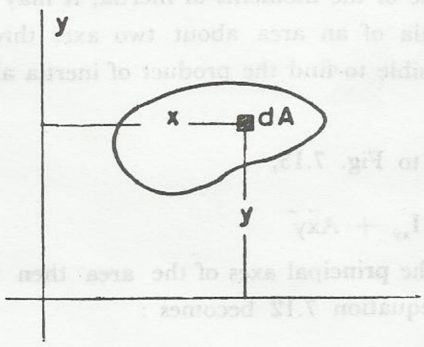

Fig. .7.11

It is evaluated by methods similar to those used in evaluating the moments of inertia. When both the x and y co-ordinates have similar signs, either positive or negative, the product of inertia is positive, but when the two co-ordinates have different signs, it is negative. If the area is symmetrical with respect to one of the axes as shown in Fig. 7.12, each elemental area dA on one side of the axis of symmetry will have a

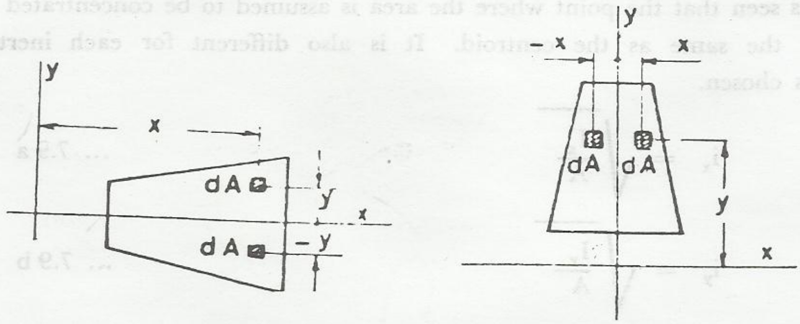

Fig. 7.12

corresponding area on the other side which while having one similar co-ordinate. the second is of opposite sign. It is obvious therefore that the sum of the products of inertia of the two elements will be zero, and consequently the product of inertia of the whole area will also be zero. Thus, it may be stated that when either one of the two centroidal axes is an axis of symmetry, the product of inertia is zero.

$$I_{xy} = 0 \qquad \dots 7.11$$

Such two axes are called principal axes of inertia.

As in the case of the moments of inertia, it may be shown that if the product of inertia of an area about two axes through the centroid is known, it is possible to find the product of inertia about any other set of parallel axes.

Thus, referring to Fig. 7.13,

$$I_{x'y'} = I_{xy} + A\bar{x}\bar{y} \qquad \dots 7.12$$

If x and y are the principal axes of the area then from equation 7.11 $I_{xy} = 0$, and equation 7.12 becomes :

$$I_{x'y'} = A\bar{x}\bar{y} \qquad \dots 7.13$$

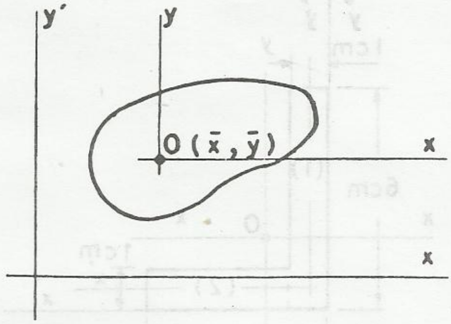

Fig. 7.13

For an area composed of symmetrical elements, the product of inertia of the entire area is obtained as the sum of the values found by using equation 7.13 for each element. The following examples will illustrate this point.

Example 7.5 Find the product of inertia I_{xy} for the Z-section shown in Fig. 7.14.

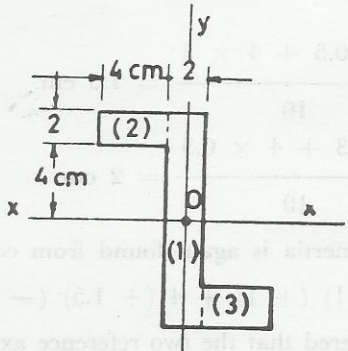

Fig. 7.14

Solution :

The total area is divided into three rectangular elements (1), (2) and (3). Rectangle (1) is symmetrical about both x and y axes, hence its product of inertia is zero. Rectangles (2) and (3) are symmetrical about axes through their centroids, therefore their product of inertia may be found from equation 7.13.

for rectangle (2), $I_{xy} = (4 \times 2)\ (-3)\ (+5) = -120 \text{ cm.}^4$

for rectangle (3), $I_{xy} = (4 \times 2)\ (+3)\ (-5) = -120 \text{ cm.}^4$

for the whole section, $I_{xy} = -120 - 120 = -240 \text{ cm.}^4$

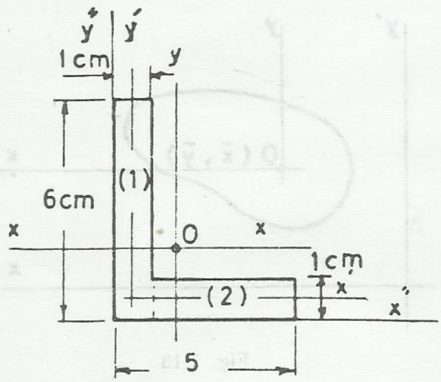

Fig. 7.15

Example 7.6 Find the product of inertia about horizontal and vertical axes through the centroid of the unequal angle shown in Fig. 7.15.

Solution : The total area is divided into two rectangular elements (1) and (2) as shown. The co-ordinates of the centroid with respect to two axes x'' and y'' chosen as the outer edges of the angle are obtained as follows :

$$x_1 = \frac{6 \times 0.5 + 4 \times 3}{10} = 1.5 \text{ cm.}$$

$$y_1 = \frac{6 \times 3 + 4 \times 0.5}{10} = 2 \text{ cm.}$$

and the product of inertia is again found from equation 7.13

$$I_{xy} = 6\ (-1)\ (+1) + 4\ (+1.5)\ (-1.5) = -15 \text{ cm.}^4$$

It should be remembered that the two reference axes x'' and y'' are arbi trarily chosen. If the reference axes were chosen as x' and y' through the centroids of the two rectangles (1) and (2), another shorter solution may be worked out.

Since rectangle (1) is symmetrical about the y' axis and rectangle (2) is symmetrical about the x' axis then $I_{x'y'} = 0$. The co-ordinates of the centroid with respect to these two axes are :

$$x_2 = \frac{4 \times 2.5}{10} = 1 \text{ cm.}$$

$$y_2 = \frac{6 \times 2.5}{10} = 1.5 \text{ cm.}$$

and the product of inertia is readily obtained from equation 7.11.

$$I_{x'y'} = 0 = I_{xy} + 10 \times 1 \times 1.5$$

$$I_{xy} = -15 \text{ cm.}^4$$

which is identical to the value obtained before.

7.8 Moments and product of inertia about inclined axes

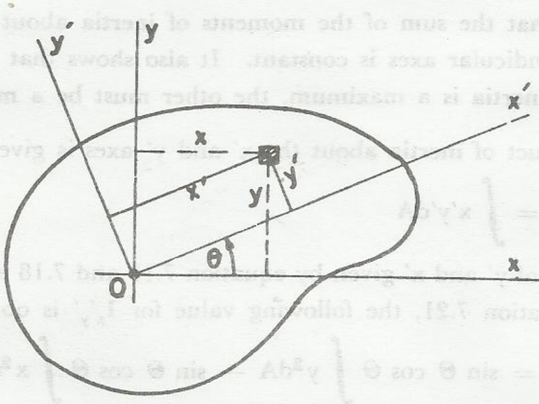

Fig. 7.16

The moments of intertia of an area about inclined axes may be obtained from the properties of the area with respect to the horizontal and vertical axes.

Referring to Fig. 7.16,

$$I_x' = \int y'^2 dA \qquad \text{... } 7.14$$

$$y' = y \cos\Theta - x \sin\Theta \qquad \text{... } 7.15$$

If this value of y' is substituted in equation 7.14, the following value for I_x' is obtained,

$$I_x' = \cos^2\Theta \int y^2 dA + \sin^2\Theta \int x^2 dA - 2 \sin\Theta \cos\Theta \int xy dA \qquad \text{... } 7.16$$

The first and second integrals of equation 7.16 represent the moments of inertia of the area about the x and y axes respectively. The last integral represents the product of inertia I_{xy}. Equation 7.16 may thus be written as :

$$I_x' = I_x \cos^2\Theta + I_y \sin^2\Theta - I_{xy} \sin 2\Theta \qquad \text{... } \mathbf{7.17a}$$

Using the trigonometric relation,

$$x' = x \cos\Theta + y \sin\Theta \qquad \text{... } 7.18$$

a similar expression may be derived for the moment of inertia about the y'-axis.

$$I_y' = I_x \sin^2\Theta + I_y \cos^2\Theta + I_{xy} \sin 2\Theta \quad \ldots 7.19\text{ a}$$

If equations 7.17 and 7.19 are added, the following relationship is obtained.

$$I_x' + I_y' = I_x + I_y \quad \ldots 7.20$$

This shows that the sum of the moments of inertia about any two centroidal perpendicular axes is constant. It also shows that if one of the moments of inertia is a maximum, the other must be a minimum.

The product of inertia about the x' and y' axes is given by :

$$I_{x'y'}' = \int x'y'dA \quad \ldots 7.21$$

If the values of y' and x' given by equation 7.15 and 7.18 are then substituted in equation 7.21, the following value for $I_{x'y'}$ is obtained.

$$I_{x'y'} = \sin\Theta\cos\Theta \int y^2dA - \sin\Theta\cos\Theta \int x^2dA + \cos^2\Theta \int xydA - \sin^2\Theta \int xydA$$

$$I_{x'y'} = (I_x - I_y) \sin\Theta\cos\Theta + I_{xy} (\cos^2\Theta - \sin^2\Theta) \ldots 7.22\text{ a}$$

If the relationships

$$\cos^2\Theta = \frac{1}{2} + \frac{1}{2}\cos 2\Theta,\ \sin^2\Theta = \frac{1}{2} - \frac{1}{2}\cos 2\Theta, \text{ and } \sin 2\Theta = 2\sin\Theta\cos\Theta$$

are substituted in equations 7.17a, 7.19a and 7.22a, the result will by :

$$I_x' = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\Theta - I_{xy}\sin 2\Theta \ldots \quad 7.17\text{ b}$$

$$I_y' = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\Theta + I_{xy}\sin 2\Theta \quad \ldots 7.19\text{ b}$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\Theta + I_{xy}\cos 2\Theta \quad \ldots 7.22\text{ b}$$

7.9 Principal axes of inertia

The principal axes of inertia of a plane area, which are usually referred to as axes u and v, may be defined as the two perpendicular axes passing through the centroid of the area such that the moment of inertia about one is a maximum and about the other a minimum, or alternatively, the two axes the product of inertia about which is zero.

As may be seen from equations 7.17 and 7.19, the moment of inertia of an area about an inclined axis is a function of the angle Θ. The angle Θ at which the moment of inertia is a maximum or a minimum is obtained by differentiating equation 7.17 with respect to Θ and equating the derivative to zero. Thus,

$$\frac{dI_x}{d\Theta} = - I_x \sin 2\Theta + I_y \sin 2\Theta - 2 I_{xy} \cos 2\Theta = 0$$

or, $$\tan 2\Theta = \frac{-2 I_{xy}}{I_x - I_y} \qquad \ldots 7.23$$

Since there are two angles under 360^0 which have the same tangent, equequation 7.23 defines two values for the angle 2Θ which are at 180^o The two corresponding values of Θ will be at 90^0. By definition, the two perpendicular axes defined by equation 7.23 are the principal axes.

By the alternative definition, if Θ is to define the principal axes, $I_{x'y'} = 0$. Thus, **from equation 7.22 b,**

$$\tan 2\Theta = \frac{-2 I_{xy}}{I_x - I_y}$$

which is the same result obtained in equation 7.23. The moments of inertia about the principal axes I_u and I_v may be obtained by substituting the values of Θ obtained from equation 7.23 into equation 7.17**b.** Thus,

$$I_u = \frac{I_x + I_y}{2} + \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I^2_{xy}} \qquad \ldots 7.24$$

$$I_v = \frac{I_x + I_y}{2} - \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I^2_{xy}} \qquad \ldots 7.25$$

where I_u and I_v are the maximum and minimum values of the moments of inertia respectively.

7.10 Semi-graphical treatment — Mohr's circle of inertia

The expressions for the moments and products of inertia about inclined axes given by equations 7.17 and 7.22 are difficult to remember. It is convenient, therefore, to use a graphical solution which is easy to remember. A careful study of equations 7.17_a and 7.22_a shows that they represent a circle written in a parametric form. That they represent a circle is made clearer by first re-writing them as :

$$I_{x\theta} - \frac{I_x + I_y}{2} = \frac{I_x - I_y}{2} \cos 2\Theta - I_{xy} \sin 2\Theta$$

$$I_{xy\theta} = \frac{I_x - I_y}{2} \sin 2\Theta + I_{xy} \cos 2\Theta$$

This is done by making use of the trigonometric relationships,

$$\sin^2 \Theta = 1/2 - 1/2 \cos 2\Theta$$

$$\cos^2 \Theta = 1/2 + 1/2 \cos 2\Theta$$

Eliminating the parameter Θ by squaring and adding,

$$\left(I_{x\theta} - \frac{I_x + I_y}{2} \right)^2 + I^2_{xy\theta} = \left(\frac{I_x - I_y}{2} \right)^2 + I^2_{xy} \quad \ldots \quad 7.26$$

However, in every problem I_x, I_y and I_{xy} are constants while $I_{x\theta}$ and $I_{xy\theta}$ are the variables. Hence equation 7.26 may be written in a simplified form,

$$(I_{x\theta} - a)^2 + I^2_{xy\theta} = b^2 \quad \ldots \quad 7.27$$

where $a = \dfrac{I_x + I_y}{2}$ ... 7.28 a

$b = \sqrt{\left(\dfrac{I_x - I_y}{2} \right)^2 + I^2_{xy}}$ ... 7.28 b

Equation 7.27 is the familiar expression known in analytical geometry; $(x-a)^2 + y^2 = b^2$ of a circle of radius b and centre at (a, 0). Hence if a circle satisfying this equation is plotted, the co-ordinates of a point (x, y) on this circle correspond to $I_{x\theta}$ and $I_{xy\theta}$ for a particular inclination Θ with respect to the reference axes x and y. The x co-ordinate represents the moment of inertia while the y co-ordinate represents the product of inertia. The circle so constructed is called ***Mohr's circle* of inertia.**

There are several methods of plotting the circle defined by equation 7.26. It may be constructed by locating the centre at (a, 0) and using the radius b given by equation 7.28 b, but this is not the best procedure for the purpose at hand. The moment of inertia and product of inertia about two centroidal rectangular axes x and y are usually known. These two values, I_x and I_{xy}, define one point on the circle. This Knowledge, together with the fact that the centre of circle is located on the abscessa at $(I_x + I_y)/2$ is sufficient to plot the circle. The procedures are outlined below with reference to Fig. 7.17.

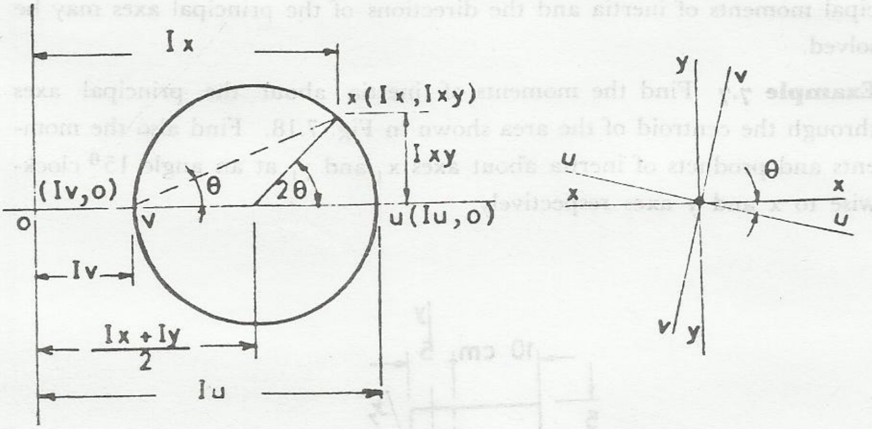

Fig. 7.17

(1) Set up a rectangular co-ordinate system of axes where the horizontal axis is the moment of inertia and the vertical axis is the product of inertia axis. Directions of positive axes are taken, as usual, to the right and upward.

(2) Locate the centre of the circle, which is on the horizontal axis at a distance of $(I_x+I_y)/2$ from the origin o

(3) Locate the point x of co-ordinates I_x and I_{xy} with respect to the origin; I_{xy} measured upwards if positive and downwards if negative.

(4) Connect the centre of the circle found in (2) with the point located in (3) and determine this distance which is the radius of the circle.

(5) Draw a circle with the radius found in (4). The two points of intersection with the horizontal axis give the values of the two principal moments of inertia I_u and I_v.

(6) The directions of the u-axis makes an angle Θ, found from the geometry of the circle, with the x-axis. A clockwise rotation of the axis to be found corresponds to a clockwise rotation round the circle. The v-axis is obtained in a similar manner and makes an angle Θ with the y-axis.

By following the above procedures in constructing Mohr's circle of inertia, the complete problem of determining the magnitude of the principal moments of inertia and the directions of the principal axes may be solved.

Example 7.7 Find the moments of inertia about the principal axes through the centroid of the area shown in Fig. 7.18. Find also the moments and products of inertia about axes x_1 and y_1 at an angle 15^0 clockwise to x and y axes respectively.

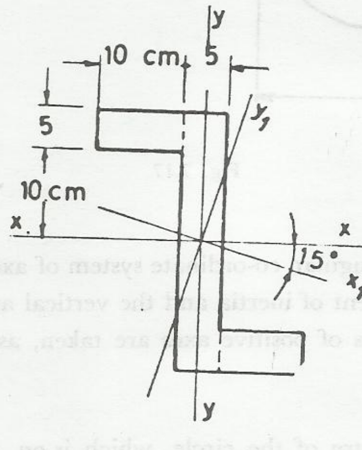

Fig. 7.18

Solution :

The moments and product of inertia about x and y axes are obtained as follows :

$$I_x = \frac{5 \times 30^3}{12} + 2 \left(\frac{10 \times 5^3}{12} + 10 \times 5 \times 12.5^2 \right) = 27098 \text{ cm.}^4$$

$$I_y = \frac{30 \times 5^3}{12} + 2 \left(\frac{5 \times 10^3}{12} + 10 \times 5 \times 7.5^2 \right) = 6774 \text{ cm.}^4$$

$$I_{xy} = (5 \times 10)(-7.5)(12.5) + (5 \times 10)(7.5)(-12.5) = -9360 \text{ cm.}^4$$

Mohr's circle may now be plotted from these three values. The centre of the circle has the co-ordinates $\left(\frac{I_x + I_y}{2}, 0\right)$ or (16936,0), while point x has the co-ordinates (I_x, I_{xy}) or (27098, — 9360) as shown in Fig. 7.19 a. Therefore,

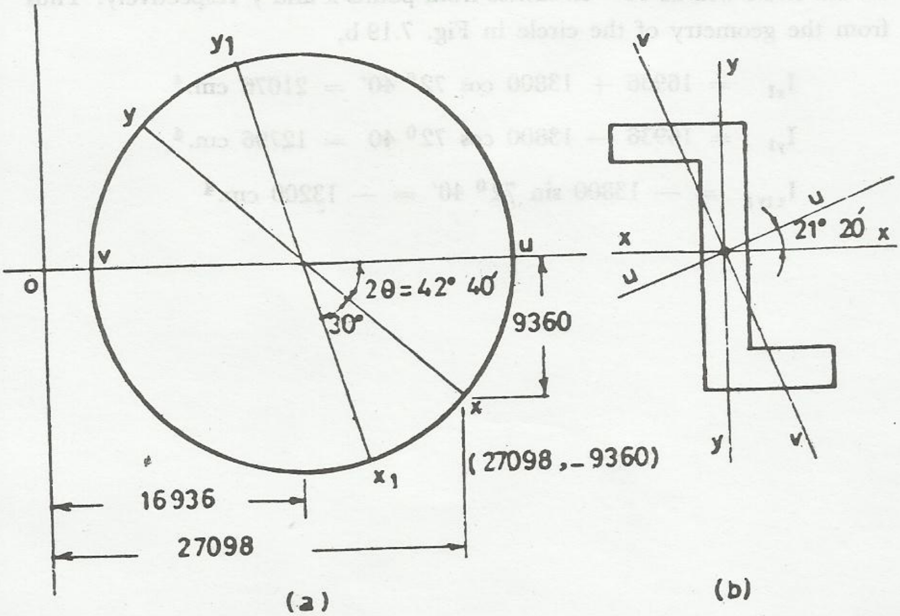

Fig. 7.19

$$\text{Radius} = \sqrt{9360^2 + 10162^2} = 13800$$

$$\tan 2\theta = \frac{9360}{10162} = 0.922$$

$$2\theta = 42^0\ 40'$$

The principal moments of inertia are equal to the distance from the origin to the centre of the circle plus or minus its radius.

$$I_u = 16936 + 13800 = 30736 \text{ cm.}^4$$

$$I_v = 16936 - 13800 = 3136 \text{ cm.}^4$$

Since point u on the circle is anticlockwise from point x, the u-axis is anticlockwise from the x-axis at an angle,

$$\theta = \frac{42^0\ 40'}{2} = 21^0\ 20'$$

The v-axis is perpendicular to the u-axis through the centroid as shown in Fig. 7.19 b.

The moments and product if inertia about the x_1 and y_1 axes are obtained from the co-ordinates of points x_1 and y_1 on the circle. Since x_1 and y_1 axes are 15° clockwise from x and y axes, the points x_1 and y_1 on the circle will be 30^0 clockwise from points x and y respectively. Thus from the geometry of the circle in Fig. 7.19 b,

$$I_{x1} = 16936 + 13800 \cos 72^0\, 40' = 21076 \text{ cm.}^4$$

$$I_{y1} = 16936 - 13800 \cos 72^0\, 40 = 12796 \text{ cm.}^4$$

$$I_{x1y1} = - 13800 \sin 72^0\, 40' = - 13200 \text{ cm.}^4$$